Conformal Field Theory and Gravity

Solutions to Problem Set 1

Fall 2024

1. The variational principle of General Relativity

(a) This follows from $\delta(\sqrt{-g}g^{\mu\nu}R_{\mu\nu}) = \delta(\sqrt{-g})R + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$ and $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$.

(b) This follows from some straightforward algebra after plugging $\delta\Gamma^{\lambda}_{\mu\nu}$ in $\delta R_{\mu\nu}$.

(c) Expanding $g_{\mu\nu} = \gamma_{\mu\nu} + \epsilon n_\mu n_\nu$, one obtains that the $n_\mu n_\nu$ terms cancel out and

$$\delta S_{\text{EH}} = \text{e.o.m.} + \frac{\epsilon}{16\pi G}\int d^{d-1}x\sqrt{\gamma}(n^\lambda\gamma^{\mu\nu}\nabla_\nu\delta g_{\mu\lambda} - n^\lambda\gamma^{\mu\nu}\nabla_\lambda\delta g_{\mu\nu}) \tag{1}$$

Using the Dirichlet boundary condition, the first term vanishes and we obtain the desired result.

(d) After straight-forward computation,

$$K_{\mu\nu} = \nabla_\mu n_\nu + \nabla_\nu n_\mu - \frac{\epsilon}{2}(n_\mu n^\rho\nabla_\rho n_\nu + n_\nu n^\rho\nabla_\rho n_\nu + n_\rho n_\nu\nabla_\mu n^\rho + n_\rho n_\mu\nabla_\nu n^\rho) \tag{2}$$

The last two terms vanish because $n_\rho\nabla_\alpha n^\rho = \frac{1}{2}\nabla_\alpha(n_\rho n^\rho) = 0$

(e) Using the hint,

$$\nabla_\mu n_\nu = (\nabla_\mu\alpha)\nabla_\nu f + \alpha\nabla_\mu\nabla_\nu f = \frac{1}{\alpha}n_\nu\nabla_\mu\alpha + \nabla_\mu\nabla_\nu f \tag{3}$$

Thus,

$$\nabla_\mu n_\nu - \nabla_\nu n_\mu = \frac{1}{\alpha}\left(n_\nu\nabla_\mu\alpha - n_\mu\nabla_\nu\alpha\right) \tag{4}$$

Also, using $n_\mu = \alpha\nabla_\mu f$,

$$\begin{aligned} n^\lambda n_\nu\nabla_\lambda n_\mu &= n^\lambda n_\nu(\nabla_\lambda\alpha)\nabla_\mu f + \alpha n^\lambda n_\nu\nabla_\lambda\nabla_\mu f \\ &= n^\lambda n_\nu(\nabla_\lambda\alpha)\nabla_\mu f + n^\lambda n_\nu\nabla_\mu(\underbrace{\alpha\nabla_\lambda f}_{n_\lambda}) - n^\lambda n_\nu(\nabla_\mu\alpha)\nabla_\lambda f \\ &= \frac{1}{\alpha}n_\mu n_\nu n^\lambda\nabla_\lambda\alpha - \frac{1}{\alpha}\epsilon n_\nu\nabla_\mu\alpha \end{aligned} \tag{5}$$

where in going from the first to the second line we interchange the λ and μ derivatives of the second term and introduced α in the first derivative, and from the second to the third we used $n^\lambda\nabla n_\lambda = 0$ and $n^\lambda\nabla_\lambda f = \frac{1}{\alpha}n^\lambda n_\lambda = \frac{1}{\alpha}\epsilon$. Thus,

$$\epsilon(n^\lambda n_\nu\nabla_\lambda n_\mu - n^\lambda n_\mu\nabla_\lambda n_\nu) = -\frac{1}{\alpha}n_\nu\nabla_\mu\alpha + \frac{1}{\alpha}n_\mu\nabla_\nu\alpha \tag{6}$$

since $\epsilon^2 = 1$. This is precisely the opposite of (4), giving the desired result.

(f) We have

$$\begin{aligned} K = g^{\mu\nu} K_{\mu\nu} &= \nabla^\mu n_\mu - \epsilon n^\lambda n^\mu \nabla_\lambda n_\mu \\ &= g^{\mu\nu} \nabla_\nu n_\mu - \epsilon n^\mu n^\nu \nabla_\nu n_\mu = \gamma^\nu{}_\mu \nabla_\nu n^\mu \end{aligned} \tag{7}$$

With Dirichlet boundary conditions, the only varying quantity is $\nabla \sim \partial + \Gamma$. The derivative piece ∂ does not vary under metric variations, thus

$$\delta K = \gamma^\nu{}_\mu \delta\Gamma^\mu_{\nu\rho} n^\rho = \frac{1}{2} n^\rho \gamma^{\nu\tau} \nabla_\rho \delta g_{\nu\tau} \tag{8}$$

where in the last equality we used $\delta\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\tau}(\nabla_\nu \delta g_{\rho\tau} + \nabla_\rho \delta g_{\nu\tau} - \nabla_\tau \delta g_{\nu\rho})$, the first and third contribution cancelling each other.

(g) To recap, we've shown in part (c) that,

$$\delta S_{\rm EH} = \text{e.o.m.} - \frac{\epsilon}{16\pi G} \int d^{d-1}x \sqrt{\gamma} n^\lambda \gamma^{\mu\nu} \nabla_\lambda \delta g_{\mu\nu} \tag{9}$$

and in part (f) that,

$$\delta K = \frac{1}{2} n^\lambda \gamma^{\mu\nu} \nabla_\lambda \delta g_{\mu\nu} \tag{10}$$

Thus, by defining

$$S_{\rm GHY} \equiv \frac{\epsilon}{8\pi G} \int d^{d-1}x \sqrt{\gamma} K \tag{11}$$

We have that with Dirichlet boundary conditions, $\delta(\sqrt{\gamma}) = 0$ and thus,

$$\delta S_{\rm GHY} = \frac{\epsilon}{16\pi G} \int d^{d-1}x \sqrt{\gamma} n^\lambda \gamma^{\mu\nu} \nabla_\lambda \delta g_{\mu\nu} \tag{12}$$

which cures the variational principle of GR,

$$\delta(S_{\rm EH} + S_{\rm GHY})\Big|_{\delta g_{\mu\nu}=0 \text{ on the boundary}} = \text{e.o.m.} \tag{13}$$

2. ADM energy

(a) On S_r^2 we will use capital indices we have the metric $\sigma_{AB} = \text{diag}(r^2, r^2\sin^2(\theta))$. The normal vector at any point, living in the 3D space Σ_t is

$$\sigma^i = (1 - \frac{2M}{r})^{\frac{1}{2}}(\partial_r)^i = (1 - \frac{M}{r} + \mathcal{O}(r^{-2}))(\partial_r)^i \tag{14}$$

Here capital indices denote objects living on S_r^2, while normal latin indices denote objects living on Σ_t Therefore, the extrinsic curvature k_S is

$$k_S = \frac{1}{2}\sigma^{AB}\mathcal{L}_\sigma(\sigma_{AB}) = \frac{1}{2}\sigma^{AB}\sigma^r\partial_r\sigma_{AB} = \frac{2}{r}(1 - \frac{M}{r}) \tag{15}$$

Subtracting the contribution from Minkowski space, we get $k_S - k_S^0 = -\frac{2M}{r^2}$, and we can compute the integral

$$E_{ADM} = -\frac{1}{8\pi}\lim_{r\to\infty}(4\pi r^2)(-\frac{2M}{r^2}) = M \tag{16}$$

The term containing the momentum π^{ij} can be neglected since N_i is exactly zero in these coordinates.

(b) From the first hint, we get $j^a = \nabla_b\nabla^a k^b$.
Conservation follows from the second hint and the fact that $\nabla^a k^b$ is antisymmetric due to the Killing equation $\nabla^{(a}k^{b)} = 0$. Indeed

$$\nabla_a j^a = \nabla_a\nabla_b\nabla^a k^b = 0 \tag{17}$$

(c) The Komar mass is

$$\begin{aligned} E_k[\Sigma] =& \frac{1}{4\pi}\int_\Sigma \sqrt{h}\, n^a\nabla_b\nabla_a k^b \\ =& \frac{1}{4\pi}\int_{\partial\Sigma} \sqrt{\sigma}\, n^a\sigma^b\nabla_a k_b \end{aligned} \tag{18}$$

where in the second line we used Stoke's theorem on Σ.
On a curved spacetime, the a timelike Killing vector generalises the usual concept of time translation generator. j^a represents the associated Noether current and E_k the associated conserved charge, i.e. energy.
In fact, the Noether current in flat space is simply $j'_a = T_{ab}k^b$, where T_{ab} is the stress tensor. Now, if we use Einstein's equations, we get

$$j'_a = \frac{1}{8\pi}(R_{ab} - \frac{1}{2}Rg_{ab})k^b = \frac{1}{8\pi}(j_a - \frac{1}{2}Rk_a) \tag{19}$$

The quantity only differ by a term which is zero in flat space and does not give rise to any boundary term, hence the currents are equivalent in flat space.
Important comment: while in this exercise we used the fact that k^a is a timelike Killing vector everywhere, one can relax this assumption by requiring that k^a is a Killing vector only at spatial infinity. The formula for the Komar mass 18 can then still be used as it only relies on the asymptotic data.

(d) Using $n = k = \partial_t$, $\sigma = \partial_r$, the formula above gives

$$\begin{aligned} E_k =& \frac{1}{4\pi}\int_{\partial\Sigma} \sqrt{\sigma}\; n^a \sigma^b \nabla_a k_b \\ =& \frac{1}{4\pi}\lim_{r\to\infty}(4\pi r^2)(\Gamma^r_{tt}) = \lim_{r\to\infty}(r^2)\frac{M}{r^2} \\ =& \, M \end{aligned} \tag{20}$$

(e) Spacetimes with compact spatial slices have no boundary, hence E_{ADM} cannot be defined. Important examples are de Sitter space and all FRW cosmological spacetimes with $K = +1$, since global spatial slices are 3-spheres.

(f) It is only possible to define the Komar mass if the spacetime has a timelike Killing vector, i.e. it is stationary. Of course, most spacetimes are not stationary (e.g. expanding cosmological spacetimes), so energy cannot always be defined in this way.